

Information Retention by Stringy Black Holes

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Building upon our previous work on two-dimensional stringy black holes and its extension to spherically-symmetric four-dimensional stringy black holes, we show how the latter retain information. A key rôle is played by an infinite-dimensional W_∞ symmetry that preserves the area of an isolated black-hole horizon and hence its entropy. The exactly-marginal conformal world-sheet operator representing a massless stringy particle interacting with the black hole necessarily includes a contribution from W_∞ generators in its vertex function. This admixture manifests the transfer of information between the string black hole and external particles. We discuss different manifestations of W_∞ symmetry in black-hole physics and the connections between them.

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1. Introduction

The black-hole information problem requires no introduction. Decades ago, Bekenstein [1] and Hawking [2] discovered that four-dimensional black holes have thermodynamical properties such as temperature and non-zero entropy corresponding to a mixed quantum-mechanical state. Hawking, in particular, then argued [3] that information would be lost across the black-hole horizon, giving rise to a transition from a pure to a mixed state.

The advent of string theory, and in particular Witten's construction [4] of a two-dimensional black hole solution using an $SU(1,1)/U(1)$ coset structure [5, 6], coupled with dualities [7] and followed by the construction of four-dimensional stringy black holes using D-branes [8, 9], provided opportunities to probe the black-hole information problem in an explicit theoretical laboratory. We have argued [10] that two-dimensional black holes carry an infinite set of 'hairy' W quantum numbers that preserve *in principle* the lost information, though *in practice* this information is inaccessible. We have also argued that these observations can be extended to spherically-symmetric four-dimensional black holes [12], whose horizon geometry is encoded in a similar $SU(1,1)/U(1)$ coset structure, accompanied by a similar infinite-dimensional W symmetry and an associated infinite set of 'hair' that is measurable in principle [13].

Constructions using D-branes provided explicit examples of four-dimensional black-hole solutions whose microstates could be counted [8], giving numbers consistent with the Bekenstein-Hawking entropy and suggesting that indeed the 'lost' information could *in principle* be retained. However, there still remained the issues how the information was transferred to and from these microstates, in what form it was stored, and whether the information transferred into the microstates could *in practice* be extracted, or whether it was in reality lost.

An interesting approach to addressing these issues has recently been taken in a series of papers by Strominger and collaborators [14, 15]. They have shown that spherically-symmetric four-dimensional black holes carry an infinite set of gravitational 'hair' associated with BBMS supertranslations [16, 17] on the retarded null infinity \mathcal{I}^+ , corresponding to vacua that differ by the addition of soft gravitons and could be measured via the gravitational memory effect. They also found an infinite set of inequivalent electromagnetic gauge configurations corresponding to electromagnetic hair and differing by the addition of soft photons. It has recently been suggested by Hawking [18] that the apparent information loss paradox might be resolved by considering supertranslations on the horizon, with the information 'lost' by incoming particles being recoverable *in principle*, though lost *in practice*.

This proposal raises many questions, including whether supertranslations (together with superrotations and the corresponding electromagnetic gauge configurations) are capable of encoding *all* the information carried by the incoming particles [19], as well as the relation to the stringy description of black holes, the details of the mechanism for information transfer to and from the black hole, and whether information is really lost. We mention here for completeness the recent work of 't Hooft [20], where it is argued that the black-hole horizon should be viewed as an apparent world-sheet of an induced string theory. This is to be contrasted with the spirit of our approach in this paper, where we start from string theory at a fundamental level. We also mention the recent work of Polchinski [21] in which a shock-wave approximation was used to calculate the shift on a generator of the horizon caused by an ingoing wave packet, which is similar in spirit to our analysis

below of the supertranslation of the horizon (viewed as a recoiling D-brane) induced by infalling matter. Other works in similar spirit include [22, 23], where the back-reaction of matter falling onto the black hole horizon (and fluctuations of the latter) is argued to play an important rôle in retaining information.

In this paper we review our previous arguments about the importance of W_∞ symmetry [24, 25]¹ for ‘balancing the information books’. In particular, we recall that a massless stringy particle interacting with the black hole is represented by a conformal operator on the world-sheet of the string, which is exactly marginal *only if* a contribution from W_∞ generators is included in its vertex function [10, 6]. Without this contribution, the corresponding renormalization-group (RG) β function is non-zero, leading to an inexorable increase in entropy. As we recall [10, 11], w_∞ (which is the classical limit of the quantum W_∞ symmetry) is the algebra of transformations that preserve the two-dimensional phase-space volume of massless (‘tachyonic’) stringy matter propagating in the background of a stringy black hole.

Moreover, these quantum W -algebras are symmetries of the quantum scattering matrix of the corresponding two-dimensional string theory [26, 6], in the sense that the operator product expansion between two appropriate vertex operators reproduce the corresponding W -algebra. In the flat space-time case (in which case the string theory is just a two-dimensional Liouville theory) the operators corresponding to the discrete higher-spin operators of the W algebra are discretized ‘tachyon’ operators. However, as already mentioned, in the presence of a black hole, at the quantum level, the corresponding W_∞ symmetries necessarily mix massless and massive (topological, delocalised) stringy states [6]. The admixture of W_∞ generators in the exactly marginal vertex operator of a massless string excitation shows how information is transferred between a stringy black hole and external particles.

We discuss below the embedding of the two-dimensional coset describing the singularity in a four-dimensional space-time [12] with the structure $SU(1,1)/U(1) \otimes S^2$, where S^2 is a two-dimensional manifold with the topology of the sphere that is to be identified with the horizon of the four-dimensional black hole. Under certain circumstances specified below, in addition to the quantum W -symmetries that leave the (quantum-gravity) scattering matrix invariant, and are associated with the discrete (topological) states of the two-dimensional coset substructure, there is also a classical w_∞ algebra of symmetry transformations of the horizon coordinates that preserves the area of the horizon of an isolated spherically-symmetric four-dimensional black hole, so that its entropy (which is known to be a Noether charge [27]) is conserved. The precise relation between these two different W -symmetries is not yet completely understood. It becomes evident, however, in the case where the horizon of the black hole is represented as a thick (recoiling) brane, which is known to correspond to an $SU(\infty)$ gauge theory [28, 29]. The black hole emerging in the low-energy limit of this string/brane theory is the *infinitely-coloured* $SU(\infty)$ black hole discussed in [30], which is reviewed below. In this case, the gauge states on the horizon can be represented as open-string states whose ends are attached to the horizon brane, which then carry the $SU(\infty)$ charges. It is known that classically such $SU(\infty)$ symmetries are isomorphic to the w_∞ algebra that preserves a two-dimensional area, which can in this case can be identified with the horizon

¹This becomes $W_{1+\infty}$, if one includes conformal spin-one states. We denote by lower-case w_∞ the classical symmetry, and by upper-case $W_{1+\infty}$ its quantum counterpart, which includes an additional conformal spin-one state.

area of the spherically-symmetric, infinitely-coloured black hole.

2. Entropy Increase and Non-Critical String

In the usual treatment of critical string theory, entropy remains constant and, in particular, pure initial states remain pure during the time evolution. A stringy discussion of the possibility of information loss therefore requires going beyond the critical string framework. Accordingly, we now recall our arguments [11] on entropy increase in non-critical string theory, *i.e.*, in a string model where conformal invariance on the world-sheet surface Σ is broken by relevant operators.

In such a case, the world-sheet dynamics depends on interaction terms that represent non-critical deformations of the form

$$S_\sigma = S_\sigma^* + \int_\Sigma d^2\xi \sqrt{-\gamma} g^i V_i \quad (2.1)$$

where S_σ^* is a conformal fixed-point σ -model action, summation over repeated indices is implied, γ is a world-sheet metric, and the set of $\{g^i\}$ is an (infinite in general) set of target-space fields associated with the corresponding vertex operators V_i . These target-space fields include the lowest-lying (massless) string modes, *i.e.*, the graviton, dilaton and antisymmetric tensor fields, the scalar ‘tachyons’ (in the case of a non-supersymmetric target space), as well as the infinity of higher-spin states ².

We introduce a global world-sheet RG scale μ , and define $\mathcal{T} \equiv \ln \mu$. We consider a density matrix $\rho(g^i)$ describing the propagation of a string representing a matter state in the deformed target-space background corresponding to the world-sheet action (2.1). This density matrix is a generic function of the background fields $\{g^i\}$. The renormalizability of the world-sheet two-dimensional theory implies that any explicit dependence of $\rho(g^i)$ on \mathcal{T} , represented by the corresponding partial derivatives, is compensated by the ‘running’ of the renormalised g^i with \mathcal{T} , leading to the following world-sheet RG equation:

$$0 = \frac{d}{d\mathcal{T}} \rho(g^i) = \frac{\partial}{\partial \mathcal{T}} \rho(g^i) + \frac{dg^j}{d\mathcal{T}} \frac{\partial \rho}{\partial g^j} . \quad (2.2)$$

It should be understood that the $\partial_i \equiv \frac{\partial}{\partial g^i}$ denote functional derivatives $\frac{\delta}{\delta g^i}$ with respect to the corresponding fields g^i in target space-time. Roughly speaking,

$$\frac{dg^j}{d\mathcal{T}} \equiv \beta^j \quad (2.3)$$

is a RG β function for the ‘coupling’ g^i of the two-dimensional world-sheet field theory ³.

²In a two target-space-dimensional setting, the only propagating multiplet consists of massless scalar fields (misleadingly called ‘tachyons’), whereas the graviton and higher-spin multiplets are topological states with discrete momenta. As we discuss later, such topological states exist also in higher-dimensional target space-times, so their presence is rather generic.

³More strictly speaking, in string theory the target-space dependences of the ‘couplings’ g^i imply some diffeomorphism variations, which lead to the replacement of the corresponding β^i RG functions by the corresponding Weyl anomaly coefficients, but such complications are not relevant for our main arguments below, so we omit them here. For details see [10].

Representing (in a heuristic way) the density matrix by its Gibbs (equilibrium) form in target space, one has, in terms of the effective target-space string Hamiltonian

$$\rho(g^i) = \text{Tr} e^{-\beta H}, \quad (2.4)$$

where β is an effective ‘temperature’ in target space, *e.g.*, the Hawking temperature in the case of a black hole. We assume that the Tr operation commutes with $\partial/\partial \mathcal{T}$, which is consistent with the interpretation of the RG scale \mathcal{T} as time (see below). We then deduce that the von Neumann (entanglement) entropy,

$$\mathcal{S} \equiv -\text{Tr}(\rho \ln \rho) \quad (2.5)$$

varies with \mathcal{T} as

$$\frac{\partial \mathcal{S}}{\partial \mathcal{T}} = -\text{Tr}\left[\frac{\partial \rho}{\partial \mathcal{T}} \ln \rho\right] - \text{Tr}\left[\frac{\partial \rho}{\partial \mathcal{T}}\right]. \quad (2.6)$$

In [11] we have used Liouville dressing to restore conformal invariance on the world sheet, and argued that the Liouville mode may be regarded as a local (on the world-sheet) covariant RG scale, $\rho(\sigma)$, where σ denotes the world-sheet coordinates.

In the presence of such a local world-sheet RG scale, there are counterterms in the σ -model action of the form [11]:

$$\int_{\Sigma} \partial_{\alpha} g^i \mathcal{G}_{ij} \partial_{\alpha} g^j, \quad (2.7)$$

where $\alpha = 1, 2$ span the world-sheet coordinates. The only dependences on them of the ‘renormalised’ couplings g^i occur through their dependences on the local RG scale $\rho(\sigma)$, so we can write (2.7) as

$$\int_{\Sigma} \partial_{\alpha} \rho \partial_{\alpha} \rho \hat{\beta}^i \mathcal{G}_{ij} \hat{\beta}^j, \quad (2.8)$$

where $\hat{\beta}^i \equiv dg^i/d\rho$ is a Weyl-anomaly coefficient. *i.e.*, a world-sheet RG β -function with respect to the local RG scale. The quantity \mathcal{G}^{ij} in (2.7, 2.8) acts as a Zamolodchikov ‘metric’ in the theory space $\{g^j\}$ of the σ -model, *i.e.*, in the space of target-space background fields. From conformal-field theory considerations [11] that we do not discuss here, one has

$$\frac{d}{d\rho} Q^2[g] \propto \hat{\beta}^i \mathcal{G}_{ij} \hat{\beta}^j, \quad (2.9)$$

where $Q^2 = C[g] - c^*$ is the central-charge deficit of the corresponding non-critical string theory with central charge $C[g]$ that is a functional of $\{g^i\}$, and the quantity c^* is the central charge at a conformal point.

The local RG scale ρ plays the rôle of a Liouville mode, which dresses the renormalised couplings g^i in such a way so as to restore criticality in the $D + 1$ dimensional target space: we find from (2.8) that the scale ρ is a propagating σ -model scalar field, and thus its zero mode on the world-sheet may be interpreted as an extra target-space-time coordinate. The sign of Q^2 depends whether the theory is subcritical or supercritical, with the supercritical case corresponding to $Q^2 > 0$. Actually, as discussed in detail in [11], the dependence of $Q^2[g]$ on the local RG scale ρ is such that the derivative with respect to the world-sheet zero mode of ρ , ρ_0 , which is also identified with the global RG scale $\mathcal{T} = \rho_0$, obeys $dQ^2/d\rho_0 \propto -Q^2 + \dots$, as a result of the fact that Q^2 is proportional to the σ -model partition function, in which the RG scale couples to the

world-sheet curvature as $\int D\rho e^{-\int_{\Sigma} \rho R^{(2)}}$, where $\int_{\Sigma} R^{(2)}$ is the Euler characteristic of the world-sheet manifold. In this way, one recovers from (2.8, 2.9) via perturbation theory the kinetic term of the Liouville action, *i.e.*, a term in the σ -model-field-theory action S_{σ} of the form

$$S_{\sigma} \ni - \int_{\Sigma} \partial_{\alpha} \rho \partial_{\alpha} \rho (Q^2 \hat{+} \dots) \quad (2.10)$$

where, in our convention [11], the corresponding kinetic terms of the spatial target coordinates of the string X^I , $I = 1, \dots, 3$ have a minus sign.

In this case the zero mode of the world-sheet RG flow of this super-critical string can be identified (up to a sign) with the temporal flow in the target space-time [11]:

$$t = -\mathcal{T} (= -\rho_0) . \quad (2.11)$$

With this identification, the second term on the right-hand side of (2.6) would vanish in a theory with energy conservation on the average, as for quantum black holes.

Then, using (2.2), (2.3) and (2.11), we obtain from (2.6):

$$\frac{\partial \mathcal{S}}{\partial t} = -\text{Tr} \left[\frac{\partial \rho}{\partial t} \ln \rho \right] = \text{Tr} \left[\beta \frac{\partial H}{\partial t} \rho \ln \rho \right] = -\text{Tr} \left[\beta \beta^i \frac{\partial H}{\partial g^i} \rho \ln \rho \right] , \quad (2.12)$$

where we have used (2.4) and the fact that $H(g^i)$ is a functional of the background fields.

In a string theory setting, the effective Hamiltonian H may be identified (up to a sign) with the effective action Γ , and it is known in this case that its field variations are proportional off-shell to the β^i (which is equivalent to the well-known statement that the string conformal invariance conditions on the world-sheet are equivalent on-shell to the equations of motion of the target-space effective action):

$$\frac{\partial H}{\partial g^i} = -\frac{\partial \Gamma}{\partial g^i} = -\mathcal{G}_{ij} \beta^j , \quad (2.13)$$

where \mathcal{G}_{ij} is the Zamolodchikov metric in the space of string models [11]. From (2.12), (2.13) we then obtain

$$\frac{\partial \mathcal{S}}{\partial t} = \beta \beta^i \mathcal{G}_{ij} \beta^j \mathcal{S} . \quad (2.14)$$

Within the context of a unitary world-sheet σ -model, corresponding to a Euclidean target space-time as can be used to represent a finite-temperature black holes, the factor $\beta^i \mathcal{G}_{ij} \beta^j > 0$. In this case, then, (2.14) implies a *monotonic* entropy increase of the (positive) entropy $\mathcal{S} > 0$ as time increases, *i.e.*, during the evolution from infrared to ultraviolet on the world sheet, whenever a string propagates in a non-conformal background.

We use this result in the following, in the specific context of strings propagating in target-space black hole backgrounds.

3. W_{∞} Symmetry Retains Information in a Two-Dimensional Stringy Black Hole

The prototypical stringy black hole solution in two dimensions [4] is characterized by a world-sheet Wess-Zumino-Witten (WZW) σ -model formulated on the coset space $SL(2, R)_k / U(1)$, where

$k = 9/4$ is the Kac-Moody algebra level. The conformal invariance condition for this world-sheet σ -model induces a target-space metric corresponding to a Euclidean black hole background

$$ds^2 = dr^2 + \tanh^2 r d\tilde{\theta}^2, \quad (3.1)$$

where (r, θ) are two-dimensional coordinates, r being the radial coordinate and $\tilde{\theta}$ a compact ‘angular’ coordinate that plays the rôle of an external temperature variable: it should not be confused with a four-dimensional angular variable. The space time (3.1) looks like a semi-infinite cigar, and may be elevated to four dimensions via a similar formula with additional angular variables, as described in Section 4.

The spectrum of stringy excitations in the two-dimensional stringy black hole includes background massive topological states that possess a quantum $W_{1+\infty}$ symmetry [24, 6]⁴. This leads to an infinity of conserved charges (‘hair’) for the black hole system, rendering it completely integrable [4]. The $W_{1+\infty}$ symmetry corresponds classically to a classical infinite-dimensional w_∞ algebra of diffeomorphisms that preserve a two-dimensional area form, which could correspond to the surface of a sphere S^2 or some other two-dimensional manifold. In the case of the two-dimensional stringy black hole examined in this Section, this is a symplectic phase space ‘area’ form, which corresponds to the Hamiltonian of a particle system in the near-horizon geometry of the black hole. As we discuss later, this algebra appears also in the case of four-dimensional stringy/brany solutions interpolating between black holes and AdS spaces.

Generically, a symplectic area two-form Ω corresponding to coordinates x, y :

$$\Omega = dy \wedge dx \quad (3.2)$$

is invariant under classical symmetry transformations that leave it invariant. These area-preserving diffeomorphisms are generated by the quantities

$$v_m^\ell = y^{\ell+1} x^{\ell+m+1} \quad (3.3)$$

where ℓ and m are integers. The Poisson brackets of these generators satisfy the classical w_∞ algebra

$$\{v_m^\ell, v_{m'}^{\ell'}\} = [m(\ell' + 1) - m'(\ell + 1)] v_{m+m'}^{\ell+\ell'}. \quad (3.4)$$

This includes a Virasoro symmetry generated by the operators $L_n = v_n^0$, whose Poisson brackets obey the algebra

$$\{L_n, L_m\} = (m - n) L_{m+n}, \quad (3.5)$$

which is a subalgebra of the w_∞ algebra (3.4).

We suggested in [10] that the infinite set of charges appearing in the quantum version of the w_∞ symmetry should be considered as an infinite set of discrete hair (termed W-hair) that is responsible for the maintenance of quantum coherence for the two-dimensional stringy black hole, since the corresponding quantum-gravity scattering matrix, obtained from correlation functions of marginal world-sheet vertex operators, is invariant under these symmetries.

⁴These symmetries were first discovered in operator product expansions of vertex operators corresponding to the discrete stringy ‘tachyon’ states of the two-dimensional (target space) $c = 1$ Liouville string theory [26] that, from a target space-time point of view, is the asymptotic limit of the Euclidean black hole of [4].

Consider, for example, the propagation of a ‘tachyon’, which is a massless particle in two dimensions. In flat space it is associated with the vertex operator:

$$\phi_{-1/2,0,0}^{c,-c} = (g_{++}g_{--})^{-\frac{1}{2}} F\left(\frac{1}{2}; \frac{1}{2}; 1; \frac{g_{+-}g_{-+}}{g_{++}g_{--}}\right), \quad (3.6)$$

where F denotes a hypergeometric function and g_{ab} , $a, b = +, -$ represent the components of a generic $SL(2, R)$ element. The operator (3.6) is exactly marginal in a two-dimensional flat-space string theory.

However, this is not the case in the background of a two-dimensional space-time black hole (Euclidean or Minkowski, the latter being obtained by analytic continuation of the compact variable (‘temperature’) in the cigar metric of [4]). In this case, the corresponding *exactly marginal* operator is [6]

$$L_0^1 \bar{L}_0^{-1} = \phi_{-1/2,0,0}^{c,-c} + i(\psi^{++} - \psi^{--}) + \dots \quad (3.7)$$

where

$$\psi^{\pm\pm} \equiv: (\bar{J}^{\pm})^N (J^{\pm})^N (g_{\pm\pm})^{j+m-N}: \quad (3.8)$$

with $J^{\pm} \equiv (k-2)(g_{\pm\mp} \partial_z g_{\pm\pm} - g_{\pm\pm} \partial_z g_{\pm\mp})$, and $\bar{J}^{\pm} \equiv (k-2)(g_{\mp\pm} \partial_{\bar{z}} g_{\pm\pm} - g_{\pm\pm} \partial_{\bar{z}} g_{\mp\pm})$, where k is the WZ model level parameter [4]. The combination $\psi^{++} - \psi^{--}$ generates a level-one massive string mode, and the dots in equation (3.7) represent operators that generate higher-level massive string states⁵. As discussed in [10], these modes are solitonic, with fixed energy and momentum. As such, they are completely de-localized in space-time.

Since the flat-space ‘tachyon’ vertex operator (3.6) is not exactly marginal in a black-hole background, the corresponding RG β function is non-vanishing and hence, by (2.14), the entropy associated with tachyonic ‘matter’ increases inexorably, *i.e.*, information is lost, *if the higher-level string modes in (3.7) are neglected*. Conversely, if these string modes are taken into account, the corresponding RG β function vanishes, entropy does not increase with the world-sheet RG flow, which we identify with the target-space temporal time flow in our approach. Thus, there is no information loss: it is stored by the higher-level string modes.

In order to guarantee the exact marginality of the corresponding vertex operator (3.6), topological states must be included in the scattering matrix of strings in a two-dimensional black-hole background. These topological modes are not detectable in a local scattering experiment, leading to an apparent ‘loss’ of quantum coherence, which is an artefact of the phenomenological truncation of the scattering process within a local effective field theory (LEFT) framework. Associated with this apparent ‘loss’ of quantum coherence there is an apparent ‘increase’ in entropy at a rate quantified by the right-hand-side of (2.14), since the *truncated* RG β^i functions of the non-marginal propagating modes do not vanish.

Nevertheless, the conserved W-hair charges are in principle measurable, and ways for doing so in principle have been outlined in [12]. These are reviewed in the next Section, where we also present arguments for the elevation of the W-hair to four-dimensional space times. In this case,

⁵ Another example of an exactly-marginal operator is $L_0^2 \bar{L}_0^{-2} = \psi^{++} + \psi^{--} + \psi^{-+} + \psi^{+-} + \dots$, which also involves in an essential way operators for massive string modes. The coupling corresponding to this world-sheet deformation of the coset model is associated with a global rescaling of the target space-metric [6], and therefore to a global constant shift of the dilaton field. Thus it produces shifts in the black hole mass [4].

the area-preserving property of the W_∞ symmetry becomes important for preserving the area of the two-dimensional surface of the black-hole horizon.

4. Elevation to Four Dimensions

We have argued in [12] that the coset singularity structure of the two-dimensional stringy black hole and generic properties of its associated discrete states have counterparts for spherically-symmetric black-hole configurations in four space-time dimensions. We now review the basic arguments supporting this conjecture, which have been reinforced by subsequent formal developments.

4.1 Embedding of the Two-Dimensional Black Hole

We consider a string theory with a spherically-symmetric gravitational background of black-hole type, which is a solution of the Einstein equations, generalised to the effective field theory derived from string theory. The metric tensor is given by an Ansatz of the form:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta + e^{W(r,t)} d\Omega^2, \quad (4.1)$$

where $W(r,t)$ is a non-singular function, $x^{\alpha,\beta}$ denote the r, t coordinates, and $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ denotes the line element on a spherical surface that does not change with time.

We remind the reader that in pure gravity all the *classical* spherically-symmetric solutions to the equations of motion obtained from higher-derivative gravitational actions with an arbitrary number of curvature tensors are *static* [31], and that a similar result holds for stringy black holes at tree level. The standard Schwarzschild solution of the spherically-symmetric four-dimensional black hole can be put in the form (4.1) by an appropriate transformation of variables.

We consider the Schwarzschild solution in Kruskal-Szekeres coordinates [32]

$$ds^2 = -\frac{32M^3}{r} e^{-\frac{r}{2M}} du dv + r^2 d\Omega^2, \quad (4.2)$$

where r is a function of u, v , given by

$$\left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}} = -uv. \quad (4.3)$$

Note that, although the two-dimensional metric components depend on the variables u, v , the black hole solution is nevertheless static. Changing variables to

$$\begin{aligned} e^{-\frac{r}{4M}} u &= u', \\ e^{-\frac{r}{4M}} v &= v' \end{aligned} \quad (4.4)$$

and taking into account the Jacobian J of the transformation of the area element $du dv$, we can put the two-dimensional metric in the form

$$g_{bh}(u', v') = \frac{e^{D(u', v')} du' dv'}{1 - u' v'}, \quad (4.5)$$

with the scale factor being given by $16M^2 e^{-\frac{r'(u', v')}{2M}} J(u', v')$, where r' is the coordinate r re-expressed in terms of the coordinates u', v' .

The metric (4.5) is a conformally-rescaled form of Witten's two-dimensional black hole solution [4]. Since the latter is described by an exact conformal field theory, the same is true after this conformal rescaling. From a σ -model point of view, this rescaling simply expresses a change of renormalisation scheme ⁶. The global properties, such as singularities, remain unchanged from the two-dimensional string case.

4.2 Discrete Topological States

In particular, the infinite-dimensional W-symmetry associated with the $SU(1,1)/U(1)$ coset structure of the dilaton-graviton sector in the two-dimensional model is elevated to become a model-independent feature of spherically-symmetric four-dimensional string configurations. Such structures are intimately connected with the existence of topological solitonic non-propagating states. These states are essentially spherically-symmetric solutions of the low-energy equations of motion obtained from the string theory in manifolds with topology $SU(1,1)/U(1) \times \mathcal{M}^2$, where \mathcal{M}^2 is a two-dimensional manifold of constant curvature. They are associated with jumps in the number of degrees of freedom at discrete values of energy and momentum as a result of relaxation of certain gauge theory constraints, as shown below. The simplest example is where $\mathcal{M}^2 = S^2$, the sphere, which describes the spherically-symmetric four-dimensional black hole solution of interest to us here. The associated infinity of discrete topological (non-propagating) states, with definite energies and momenta, couple to the massless propagating 'tachyon' string matter and thereby ensure conformal invariance of the associated σ -model action, as described above for the purely two-dimensional stringy black hole of [4].

The infinity of discrete topological states in a D -dimensional target-space string theory are similar in nature to those of the two-dimensional case [33, 26]. These states can be seen via the gauge conditions for a rank- n tensor multiplet:

$$D^{\mu_1} A_{\mu_1 \mu_2 \dots \mu_n} = 0, \quad (4.6)$$

where D_μ is a (curved-space) covariant derivative. To illustrate our arguments, consider the simplified case of weak gravitational perturbations around flat space, with a linear dilaton field of the form $\Phi(X) = Q_\mu X^\mu$, in which case the Fourier transform of (4.6) is

$$(p + Q)^{\mu_1} \tilde{A}(k)_{\mu_1 \mu_2 \dots \mu_n} = 0 \quad (4.7)$$

We observe that there is a jump in the number of degrees of freedom at the discrete momentum $p = -Q$. The fixed momentum corresponds to complete uncertainty in space, so such states are delocalised, and can be considered as quasi-topological and non-propagating soliton-like states. In ordinary string theories, such states carry a small statistical weight, due to the continuous spectrum of the various string modes. However, when strings propagate in spherically-symmetric four-dimensional background space-times, these discrete states assume particular importance. Such backgrounds are effectively two-dimensional, and therefore all the transverse modes of higher-rank tensors can be gauged away using Ward identities of the form (4.6), except for the *topological*

⁶The function $D(u, v)$ can be regarded also as a part of the two-dimensional dilaton in the given renormalisation scheme.

modes. In the case of four-dimensional spherically-symmetric black holes, these s -wave topological modes constitute the final stages of their evaporation [10], and assume responsibility for the maintenance of quantum coherence [10, 12].

4.3 Phase-Space-Area-Preserving w_∞ Symmetries

In another example [9], a w_∞ symmetry arises in the phase space of matter in a four-dimensional extremal solitonic black hole background in the context of $N = 2$, $D = 4$ supergravity. This is a BPS solution that interpolates between a maximally-supersymmetric AdS_4 space-time at large radial distances and $\text{AdS}_2 \times H^2$, where AdS_2 refers to the radial-coordinate/time part of the space-time and H^2 refers to the angular part of the space-time, which is a hyperbolic two-dimensional manifold of constant curvature. The $\text{AdS}_2 \times H^2$ geometry characterises the space-time near the horizon of the black hole. The analysis of [9] showed that the dynamics of a quantum-mechanical massive particle with non-trivial magnetic charge in the near-horizon geometry is described effectively by a one-spatial-dimensional Hamiltonian H , characterised by a w_∞ symmetry that preserves the two-dimensional phase-space area symplectic form $\Omega = dp \wedge dq - dH \wedge dt$, with q the spatial coordinate, p the canonical momentum and t the time. The energy spectrum of this particle is continuous and bounded from below: $E > 0$, but the ground state is non-normalizable, with an infrared (IR) divergence, which was regularised in [9] by putting the system in a box. The IR-regularised system is also invariant under a w_∞ that contains a Virasoro symmetry (3.5), which can be associated with the asymptotic symmetries of the AdS_2 space time, i.e., the diffeomorphisms that leave invariant the AdS_2 metric, whose quantum version includes a central extension. Such asymptotic symmetries are symmetries of the quantum-gravity scattering matrix for the full four-dimensional $\text{AdS}_2 \times H^2$ extremal black hole of [9] ⁷.

Hence, the particle system is characterised by an infinity of conserved charges of the v_m^ℓ type (3.3), in which the rôle of the x, y coordinates is played by appropriate combinations of the phase-space coordinates of the particle [9], and hence is completely integrable. From our point of view, the presence of an infinity of conserved quantities for the particle in the near-horizon geometry of the black hole also guarantees quantum coherence, in the sense that the infinity of conserved charges v_m^ℓ , which remain constant during the scattering of matter off the black-hole background, retain information during the evaporation of the latter. The situation of the coherence-preserving w_∞ algebra is exactly analogous to that preserving the phase-space area for a massless ‘tachyonic’ string matter in the two-dimensional stringy black hole - or its four-dimensional extension with topology $\text{SU}(1,1)/\text{U}(1) \times S^2$ - as discussed above.

As discussed above, the elevation of such classical phase-space-area- w_∞ symmetry algebras to fully quantum coherence-preserving algebras necessarily involves discrete topological states of the string. In two dimensions, as we have seen, the latter mix with the propagating massless matter states in order to guarantee the conformal invariance of the corresponding vertex operators in the presence of a stringy black hole background [10], and hence preserve quantum coherence according to the general arguments of Section 2.

⁷An asymptotic symmetry of the quantum-gravity scattering matrix under supertranslations of generic black hole backgrounds has been examined in [14, 15].

4.4 W-hair and Quantum Coherence

As discussed in [13], each of these discrete solitonic states can be represented as a singular gauge configuration, whose conserved W -charges can be measured in principle by generalized Aharonov-Bohm phase effects. Moreover, the topological higher-spin string states leave their imprint via selection rules in the scattering matrix, where they appear as (resonance) poles, corresponding to discrete energies and momenta and leading to certain selection rules. In the stringy black hole case, there is an infinite set of such black hole soliton states, classified by the quadratic Casimir and ‘magnetic’ quantum numbers of an internal symmetry group [13], which are excited at calculable energies and decay into distinctive combinations of light final-state particles.

The stringy scattering matrix is, in general, well defined in the presence of such black-hole backgrounds, since the world-sheet correlation functions among the appropriate exactly marginal vertex operators are unitary. This is because, as mentioned previously, in addition to the parts corresponding to the propagating string states, these operators contain an infinity of topological non-propagating states. In practice, scattering experiments in the laboratory, which involve a finite number of localised (in spacetime) particle states, cannot detect the delocalised states. Hence, from the point of view of a local low-energy observer, there would be an apparent decoherence, although this would not entail any pathologies in the full stringy theory of quantum gravity.

5. Phase-Space vs Horizon-Area-Preserving W Symmetries

We now explore the potential relation between the quantum $W_{1+\infty}$ algebras that are symmetries of the stringy quantum gravity S -matrix and the classical area-preserving symmetries that preserve the horizon area of a classical (non-evaporating) black hole and hence its entropy. This relation is subtle, and at present is not understood in its full generality, at least by the authors. Nevertheless, as we shall discuss below, by representing the horizon of the black hole as a thick D(irichlet) brane, such a relation becomes evident. As a prelude to this result, we first discuss the case of an infinitely-coloured four-dimensional black hole in a $SU(N \rightarrow \infty)$ Yang-Mills gauge theory, which has an infinite amount of gauge hair, as allowed by the ‘no-hair’ theorem [30].

5.1 Black Holes with Infinitely-Coloured Hair

As we have discussed, the classical w_∞ algebra preserves the two-dimensional area of an ‘*internal space*’ with the topology of a sphere [24, 25]. The issue is whether the ‘*internal*’ sphere can be identified with the real horizon of the spherically-symmetric four-dimensional Schwarzschild black hole. To address this question, we consider examples of four-dimensional spherically-symmetric black holes with infinitely-coloured hair, which realize explicitly a classical w_∞ . These appear in an effective field theory example of a black-hole solution in $SU(N \rightarrow \infty)$ gauge theory in a four-dimensional AdS space-time with negative cosmological constant, which plays the role of a regulator for the black-hole solution that makes it well-defined [30]. This anti-de-Sitter (AdS) regulator was given physical significance via the AdS/CFT bulk/boundary correspondence, and turns out to be physically important, as we argue below. For the present discussion, the interest of these

black holes with black holes with infinitely coloured hair is that classically there is an isomorphism between $SU(N \rightarrow \infty)$ and w_∞ [24, 28]⁸.

To develop this point, consider a unit sphere S^2 with coordinates θ, ϕ and the quantities:

$$x_1 = \sin\theta \cos\phi, \quad x_2 = \sin\theta \sin\phi, \quad x_3 = \cos\theta, \quad \text{with} \quad \sum_{i=1}^3 x_i^2 = 1. \quad (5.1)$$

The spherical harmonics $Y_{\ell m}(\theta, \phi)$ are harmonic polynomials of degree ℓ in x_i :

$$Y_{\ell m}(\theta, \phi) = \sum_{i_k=1,2,3, k=1,\dots,\ell} \alpha_{i_1\dots i_\ell}^{(m)} x_{i_1} \dots x_{i_\ell}. \quad (5.2)$$

For fixed ℓ there are $2\ell + 1$ linearly independent symmetric and traceless tensors $\alpha_{i_1\dots i_\ell}^{(m)}$, $m = -\ell, \dots, \ell$. Let us consider an $SU(2)$ subgroup of the $SU(N)$ group in the limit $N \rightarrow \infty$, generated by S_i with standard commutation relations

$$[S_i, S_j] = i\epsilon_{ijk} S_k.$$

From the standard theory of angular momentum [34, 29], we know that a representation of the $N^2 - 1$ generators of the group $SU(N)$ can be expressed as follows in terms of the S_i matrices and the α -tensors in (5.2):

$$S_{\ell, m}^{(N)} = \sum_{i_k=1,2,3, k=1,\dots,\ell} \alpha_{i_1\dots i_\ell}^{(m)} S_{i_1} \dots S_{i_\ell} \\ [S_{\ell, m}^{(N)}, S_{\ell', m'}^{(N)}] = i f_{\ell, m; \ell', m'}^{(N)} S_{\ell'', m''}^{(N)}. \quad (5.3)$$

Upon the rescaling

$$S_i \rightarrow T_i \equiv \frac{2}{N} S_i, \quad (5.4)$$

we arrive at bounded matrix elements as $N \rightarrow \infty$: $|(T_i)_b^a| \leq 1$, with the well-defined commutator algebra

$$[T_i, T_j] = i \frac{2}{N} \epsilon_{ijk} T_k \rightarrow 0, N \rightarrow \infty. \quad (5.5)$$

and the Casimir element

$$T^2 = \sum_{i=1}^3 T_i^2 = 1 - \frac{1}{N} \rightarrow 1, N \rightarrow \infty. \quad (5.6)$$

We conclude from (5.5) and (5.6) that in the $N \rightarrow \infty$ limit, a representation of the (commuting) generators of the $SU(2)$ subgroup of $SU(\infty)$ is provided by the quantities x_i in (5.1). If one considers any two functions of x_i on a spherical surface, $f(x_1, x_2, x_3), g(x_1, x_2, x_3)$, each of which can be expanded in terms of the spherical harmonics (5.2), (5.5) shows that in the limit $N \rightarrow \infty$ the corresponding matrix polynomials of the generators $f(T_1, T_2, T_3)$ and $g(T_1, T_2, T_3)$ satisfy

$$\frac{N}{2i} [f, g] \rightarrow \epsilon_{ijk} \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_k}, \quad N \rightarrow \infty. \quad (5.7)$$

⁸For $SU(N)$ gauge theories with finite N , the geometry of the corresponding space is non-commutative [29], the commutativity being restored in the limit $N \rightarrow \infty$.

Replacing the $SU(2)$ generators in (5.3) by the rescaled ones (5.4), one finds for the $N^2 - 1$ matrices $T_{\ell m}^{(N)}$:

$$\frac{N}{2i} [T_{\ell, m}^{(N)}, T_{\ell', m'}^{(N)}] \rightarrow \{Y_{\ell, m}, Y_{\ell', m'}\}, \quad N \rightarrow \infty. \quad (5.8)$$

The Poisson (classical) algebra of the spherical harmonics is known to be that of the infinite-dimensional area-preserving diffeomorphisms on the sphere $SDiff(S^2)$:

$$\begin{aligned} \{Y_{\ell, m}, Y_{\ell', m'}\} = & \frac{M(\ell + \ell' - 1, m + m')}{M(\ell, m)M(\ell', m')} (\ell' m - \ell m') Y_{\ell + \ell' - 1, m + m'} \\ & + \sum_{n=1} g_{2n}(\ell, \ell') C_{\ell, m, \ell', m'}^{\ell + \ell' - 1 - 2n, m + m'} Y_{\ell + \ell' - 1 - 2n, m + m'}, \end{aligned} \quad (5.9)$$

where the M, g_{2n} are normalization functions and the structure constants C are given in the fourth paper in [24]. This algebra is known to be isomorphic to the classical area-preserving w_∞ algebra.

Since the classical gauge fields of the $SU(N \rightarrow \infty)$ gauge theory can be expanded in the basis of the matrices $T_{\ell, m}^{(N)}$, the above considerations, and in particular (5.8), indicate that in this example of an infinitely-coloured gauge black hole, this area-preserving diffeomorphism symmetry preserves the horizon area, once we identify the ‘internal’ sphere S^2 with the actual horizon sphere of the spherically-symmetric $SU(\infty)$ black hole. In this case, the entropy of the black hole can be preserved *classically* by the w_∞ hair. If one views this $SU(\infty)$ gauge theory as a low-energy limit of some string theory then, in view of our world-sheet renormalization-group interpretation of the target time that leads to (2.14), the conservation of the classical area should correspond to the conformal invariance of the corresponding world-sheet, which guarantees the vanishing of the right-hand-side of (2.14) through the zeroes of the β^i functions of appropriate combinations of the couplings g^i . In the case at hand, the set $\{g^i\}$ consists of the graviton $G_{\mu\nu}$ and $SU(\infty)$ gauge field background modes: $A_\mu^a, a = 1 \dots \infty$.

Hence one can understand the entanglement of the massless (propagating) graviton states in this four-dimensional extremal black hole with the discrete infinity of gauge states by analogy with the entanglement of the propagating massless ‘tachyonic’ matter with the infinity of the discrete massive states in the two-dimensional black hole case, (3.6, 3.7), as follows.

Upon embedding the $SU(\infty)$ gauge theory in a string model with propagating graviton and gauge field backgrounds, one deforms the corresponding σ -model by adding to the usual graviton (spin-2) deformations the following vector deformation of the gauge field A_μ^a :

$$\mathcal{L} = \int [DX] e^{\frac{1}{2\pi\alpha'}} \int_\Sigma \sqrt{\gamma} \partial_\alpha X^\mu \partial^\alpha X^\nu G_{\mu\nu}(X) \text{Tr} \left(e^{\int_{\partial\Sigma} A_\mu^a t^a \partial_\tau X^\mu} \right). \quad (5.10)$$

Here we use the standard notation for a σ -model propagating on a world-sheet Σ with a boundary $\partial\Sigma$ (to accommodate open strings corresponding to gauge field excitations), describing the motion of a string in a target-space with coordinates X^μ and a metric background $G_{\mu\nu}(X)$. The trace Tr is over colour indices, and the t^a are the generators of the $SU(\infty)$ colour group.

The presence of an AdS background with a non-vanishing cosmological constant $\Lambda < 0$ as a regulator for the black hole of [30] implies that, on embedding such a theory in a string theory, the graviton world-sheet β -function, $\beta_{\mu\nu}^G$, is no longer zero. Indeed, a σ -model one-loop analysis, which suffices for the weak gravity in the near-horizon black-hole geometry that we consider here,

is given by the target-space Ricci tensor:

$$\beta_{\mu\nu}^G = R_{\mu\nu}, \quad (5.11)$$

which for an anti de Sitter space time reads

$$R_{\mu\nu}^G = \Lambda g_{\mu\nu} \neq 0, \Lambda < 0.$$

If this divergence were not cancelled against other background fields, so as to restore marginality of the corresponding world-sheet operator, then there would be entropy increase under the RG flow, which in our approach is identified with real-time flow (2.14), and thus information loss in the classical stationary black-hole background.

The presence of an infinity of gauge fields can provide a resolution to this problem with the infinity of zero-momentum gauge field modes $A_\mu^a, a = 1, \dots, N \rightarrow \infty$ condensing:

$$\langle \sum_{a=1}^{\infty} F_{\mu\nu}^a F^{a\mu\nu} \rangle \neq 0. \quad (5.12)$$

The infinite number of colours plays a crucial rôle in guaranteeing a macroscopic occupation of the quantum-mechanical ground state of this system, which is a prerequisite for the formation of a quantum condensate. One needs delocalised zero modes, because they are constant in space-time, and hence their condensation guarantees the space-time translational invariance of the condensate. In string theory there is an infinity of higher-order self-interactions among the gauge fields in the low energy string effective action, which can lead to the formation of such a condensate.

Coupling gauge fields and gravitons, the corresponding graviton β -function (5.11) is modified in the presence of such a condensate (5.12) to

$$\beta_{\mu\nu}^G = \Lambda g_{\mu\nu} + \frac{1}{2} V(\langle F_{\mu\nu}^a F_{\mu\nu}^a \rangle) g_{\mu\nu}, \quad (5.13)$$

where $V(\langle F_{\mu\nu}^a F_{\mu\nu}^a \rangle)$ is the scalar vacuum energy arising from the condensate. The structure on the right-hand side of (5.13) is the only one consistent with Lorentz invariance of the vacuum. The form of $V(\langle F_{\mu\nu}^a F_{\mu\nu}^a \rangle)$ is, to lowest order in the field strengths:

$$V(\langle F_{\mu\nu}^a F_{\mu\nu}^a \rangle) \propto \langle F_{\mu\nu}^a F_{\mu\nu}^a \rangle + \dots > 0, \quad (5.14)$$

where the \dots indicate higher derivative terms that are present in string theory and are essential in providing the necessary self-interactions among the non-Abelian gauge fields to guarantee the formation of condensates.

We observe from (5.14) and (5.13) that a cancellation of the right-hand-side is possible because $\Lambda < 0$ for AdS backgrounds. Thus we recover a conformal graviton background in the presence of a condensate formed by the delocalised gauge states. The latter realize a W_∞ algebra, which preserves classically the horizon area of the black hole, thus providing a picture of the area-preserving nature of the W-hair that is consistent with the conformally-invariant mixing of graviton states with the infinity of gauge states.

5.2 Horizons as ‘Thick’ D(irichlet)-Branes and $SU(\infty)$ Gauge Theory

If we represent the horizon of the four-dimensional black hole as a two-brane, then we immediately face the problem of recoil once a string matter state, represented by a closed or open string, encounters the horizon surface. If it is a closed string, it may split into two open strings (to preserve the chirality of the state), whereas if it is an open string then at least one of its ends will be attached to the horizon, causing the latter to recoil in order to conserve momentum. The recoil of the horizon induces local fluctuations on the horizon that can be studied using logarithmic conformal field theory on the world-sheet [35, 36], that have also been argued to carry information [37]. A fluctuating (recoiling) horizon may be represented (from the point of view of a low-energy observer) as a ‘thick’ D-brane stack of $N \rightarrow \infty$ concentric branes. For macroscopic black holes, with large horizons compared to the wavelength of the infalling matter, such concentric branes may be well approximated locally by a stack of parallel flat $N \rightarrow \infty$ branes. Such constructions are equivalent to $SU(N \rightarrow \infty)$ gauge theories [38], as can be seen intuitively by considering the topologically-equivalent ways ($N^2 - 1$ for $SU(N)$ gauge theory) in which an open string can be attached to a stack of N parallel D-branes.

When infalling matter crosses the horizon of such a thick-horizon-brane black hole, the recoil is described by open string excitations that carry the $SU(\infty)$ charges, leading to the infinite hair of the black hole and corresponding to the horizon-area preserving w_∞ symmetry discussed previously in Subsection 5.1. The important aspect of this example is that now the $SU(\infty)$ symmetry is also a coherence-preserving symmetry of the associated quantum-gravity scattering matrix in the presence of the $SU(\infty)$ black hole.

5.3 Two-Dimensional W_∞ Symmetries as Gauged Four-Dimensional Symmetries

In this subsection we speculate on a potential generalization of the above result to arbitrary four-dimensional space-times with space-time singularities having the (stringy-black-hole-like) structure $SU(1,1)/U(1) \otimes S^2$ or, more generally, embedding two-dimensional singularity structures admitting W_∞ symmetries into spherically-symmetric four-dimensional space-times with constant curvature, as in the example of the four-dimensional black-hole soliton of [9].

Our starting point is the construction [25] of W_∞ (and w_∞) gauge theories in terms of $(d+2)$ -dimensional local fields, where d is the dimension of space-time on which the algebras live: $d = 2$ in the case of interest to us. One can define the W_∞ quantum algebra as a commutator algebra of Hermitian operators $\xi(a, a^\dagger)$, where a, a^\dagger are the harmonic-oscillator annihilation and creation operators. One may parametrize the operators $\xi(a, a^\dagger)$ using coherent states:

$$:\xi(\hat{a}, \hat{a}^\dagger): = \int d^2z e^{-|z|^2} |z\rangle \xi(z, \bar{z}) \langle z| ,$$

where $|z\rangle = e^{\hat{a}^\dagger z} |0\rangle$, $\langle z| = \langle 0| e^{\hat{a} \bar{z}}$, $\langle z'|z\rangle = e^{\bar{z}'z}$, $\hat{a}|z\rangle = z|z\rangle$, $\langle z|\hat{a}^\dagger = \langle z|\bar{z}$, and the normalization condition is $\int d^2z e^{-|z|^2} |z\rangle \langle \bar{z}| = 1$, with $d^2z \equiv \frac{1}{\pi} \text{Re} z \text{Im} z$, and the $:\xi(\hat{a}, \hat{a}^\dagger):$ is an (anti-)normal-ordered operator, where the creation operators are always placed to the right of the annihilation operators.

The coordinates z, \bar{z} are viewed in [25] as a group-theoretical ('colour') space. Introducing a gauge potential $A_\mu(x, \hat{a}, \hat{a}^\dagger)$, where $\mu = 1, \dots, d$ is a d -dimensional space time $\{x\}$ index,

$$\hat{A}_\mu(x) \equiv A_\mu(x, \hat{a}, \hat{a}^\dagger) = \int d^2z e^{|z|^2} |z\rangle A_\mu(x, z, \bar{z}) \langle z| \quad (5.15)$$

and infinite-dimensional set of infinitesimal W_∞ gauge transformations can be introduced as follows:

$$\delta \hat{A}_\mu(x) = \partial_\mu \hat{\xi}(x) + i \left[\hat{\xi}(x), \hat{A}_\mu(x) \right], \quad \delta A_\mu(x, z, \bar{z}) = \partial_\mu \xi(x, z, \bar{z}) - \{ \{ \xi, A_\mu \} \}_{\text{Moyal}}(x, z, \bar{z}), \quad (5.16)$$

where $\{ \{ \cdot, \cdot \} \}_{\text{Moyal}}$ denotes a Moyal bracket, defined as

$$\{ \{ \xi_1, \xi_2 \} \}_{\text{Moyal}}(z, \bar{z}) \equiv i \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\partial_z^n \xi_1(z, \bar{z}) \partial_{\bar{z}}^n \xi_2(z, \bar{z}) - \partial_{\bar{z}}^n \xi_1(z, \bar{z}) \partial_z^n \xi_2(z, \bar{z}) \right). \quad (5.17)$$

In this construction, the generators of W_∞ , $\rho[\xi]$, are linear functionals of $\xi(z, \bar{z})$ and satisfy at a quantum level [25]:

$$\left[\rho[\xi_1], \rho[\xi_2] \right] = i \rho[\{ \{ \xi_1, \xi_2 \} \}_{\text{Moyal}}]. \quad (5.18)$$

The classical area-preserving w_∞ Lie algebra, obtained from W_∞ by an appropriate contraction discussed in [25], is then

$$\left[\rho[\xi_1], \rho[\xi_2] \right] = i \rho[\{ \{ \xi_1, \xi_2 \} \}_{\text{Poisson}}], \quad (5.19)$$

where $\{ \cdot, \cdot \}_{\text{Poisson}}$ denotes the (classical) Poisson bracket.

Notice that, in this representation, the W_∞ gauge fields $A_\mu(x, z, \bar{z})$ are defined in a $d+2$ -dimensional space time $\{x, z, \bar{z}\}$ with a two-dimensional 'internal' space spanned by the $\{z, \bar{z}\}$ coordinates. The Yang-Mills-type \mathcal{S} action, which is invariant under the W_∞ gauge transformations, has the form (5.16),

$$\mathcal{S} = -\frac{1}{4g^2} \int d^d x \frac{1}{4} \text{Tr} \left(\hat{\mathcal{F}}_{\mu\nu} \hat{\mathcal{F}}^{\mu\nu} \right) : \hat{\mathcal{F}}_{\mu\nu} = \partial_\mu \hat{A}_\nu(x) - \partial_\nu \hat{A}_\mu(x) - i \left[\hat{A}_\mu, \hat{A}_\nu \right], \quad (5.20)$$

where g is a coupling constant, and can be rewritten using the coherent-state representation as [25]:

$$\begin{aligned} \mathcal{S} &= -\frac{1}{4g^2} \int d^d x d^2 z \sum_{n=0}^{\infty} \frac{(-1)^n}{n} \partial_z^n \mathcal{F}_{\mu\nu}(x, z, \bar{z}) \partial_{\bar{z}}^n \mathcal{F}^{\mu\nu}(x, z, \bar{z}) : \\ \mathcal{F}_{\mu\nu} &= \partial_\mu A_\nu(x, z, \bar{z}) - \partial_\nu A_\mu(x, z, \bar{z}) + \{ \{ A_\mu, A_\nu \} \}_{\text{Moyal}}(x, z, \bar{z}). \end{aligned} \quad (5.21)$$

The reader should notice the non-local nature of the action in terms of the z, \bar{z} variables. In fact, as stressed in [25], it is this non-local nature of the action that differentiates the W_∞ from w_∞ as far as the association with the $SU(\infty)$ gauge theory is concerned. It is the spectrum of the W_∞ that can be viewed as the $N \rightarrow \infty$ limit of $SU(N)$, *not* that of the w_∞ .

Thus we see that, in analogy with the $SU(N \rightarrow \infty)$ example discussed previously, the group Trace (here over the infinite-dimensional W -algebra) is replaced by an integral over the coordinates of the two-sphere S^2 . In fact, in the action (5.21) there are no exponentially-damped $e^{-|z|^2}$ factors so, in order to have well-defined expressions upon partial integration, one must require the fields and their derivatives to vanish at $z, \bar{z} \rightarrow \pm\infty$. For the case of the sphere, there is no boundary,

and hence one is not facing this problem. One should identify the (non-compact) variables z, \bar{z} with some stereographic-projection coordinates of the S^2 . In the spirit of (5.8), then, one may expect that the infinite-dimensional W_∞ gauge symmetry indeed corresponds to an area-preserving symmetry that leaves invariant the area of S^2 , $\text{SDiff}(S^2)$.

For the stringy black hole of interest, the above considerations apply if we identify the gauged W_∞ symmetry with the $W_{1+\infty}$ of the string states discussed previously, which involves discrete de-localised states. It has been shown in [39] that, for the asymptotically-flat two-dimensional string theory, i.e., the $c = 1$ Liouville model, for each of these states one can construct ‘discrete gauge states’, at various mass string levels corresponding to tensorial gauge fields, which satisfy the same W_∞ algebra as the topological discrete states of the two-dimensional string theory [26]. These states carry the $W_{1+\infty}$ charges and thus one can gauge the W_∞ algebras by coupling them to the corresponding currents. In this way one may have an explicit realisation of the gauge transformations (5.16) for the two-dimensional string. The generalization to the curved black-hole background is a non-trivial task, since one expects a mixing of various mass levels in the exactly-marginal vertex operators describing the discrete gauge states, as with the standard topological states.

In fact there is a much simpler formal picture that describes the situation non-perturbatively in the string coupling [4, 40]. Scattering theory around the particular solution of string theory that describes the asymptotic state of the two-dimensional black hole, i.e., the $c = 1$ Liouville string in a flat two-dimensional target-space background, is known to be described by a completely integrable (and soluble) one-dimensional (quantum-mechanical) matrix model. The latter is essentially a theory of free fermionic fields $\psi(t)$ interacting with an inverted harmonic oscillator potential. The fermions depend on the Liouville dimension, which is a ‘spatial’ coordinate r in this case. The W_∞ charges in that case are described by the (infinite) set of the moments of the energy (Hamiltonian). In fact, if one considers a state with incoming fermions of energies ε_i , $i = 1, \dots, k$, the conserved charges are

$$Q_n = \sum_{i=1}^k \varepsilon_i^n, \quad (5.22)$$

which are conserved for each $n = 1, 2, \dots, \infty$, where the charge $n = 1$ is the Hamiltonian of the system.

The W_∞ symmetry algebra associated with these charges has a classical limit that is associated with the canonical transformations that preserve the free-fermion phase-space (area-preserving phase-space form), as we have discussed previously. From the detailed string theory considerations of the associated Liouville theory described previously, we know that there exist discrete gauge states that carry these charges [39, 40], i.e., the Q_n (5.22) can couple to gauge fields, which constitute elements of the gauged $W_{1+\infty}$ Lie algebras discussed above, leading to the w hair of the black hole [10]. In the context of the matrix model, the space-time dimension of the gauge field is $d = 1$. The embedding of such matrix models in four dimensions can then be done by identifying the internal colour space z, \bar{z} with the horizon surface of the four-dimensional black hole, whose asymptotic state is described by the $c=1$ Liouville string theory.

5.4 Hawking radiation from spherically-symmetric black holes and $W_{1+\infty}$ symmetries

We close this Section by mentioning some important results that provide a link between phase-space-area-preserving $W_{1+\infty}$ algebras and Hawking radiation in generic four-dimensional

black holes with spherically symmetric horizons, which aids understanding the connection between phase-space $W_{1+\infty}$ symmetries and area-preserving diffeomorphisms of the horizon in a more generic context than the $SU(\infty)$ black hole case discussed previously. It was shown in [41, 42] that moments of the Hawking radiation emitted by a generic, non-stringy spherically-symmetric black hole are connected to a $W_{1+\infty}$ algebra carried by higher-spin states, whose currents are sourced by background fields of higher-spin states, which can be identified with the discrete gauge states [39] discussed in our stringy approach [10] to black-hole singularities.

Crucial to the connection of Hawking radiation to W_∞ algebras is the effective two-dimensional conformal field theory representation [43] of the dynamics of matter fields in the near-horizon geometry of a spherically-symmetric black hole, as adopted in our approach [10]. Within this context, it is known that the quanta of Hawking radiation emitted from the horizon of a spherically-symmetric black hole break general covariance. As shown in [43], this symmetry is restored (in the sense that the corresponding gravitational anomalies in the quantum gravity path integral are cancelled) in the case of a $(1+1)$ -dimensional black body at the black-hole Hawking temperature [2]. Thus, one can represent the effective two-dimensional field theory of the Hawking radiation on the black-hole horizon as a two-dimensional field theory with an infinity of two-dimensional conformal quantum fields with a thermal spectrum, with the left movers corresponding to radially infalling matter and right movers to outgoing matter.

Schwarzschild black holes emit Hawking radiation with a Planck distribution

$$N^\pm(\omega) = \frac{1}{e^{\beta\omega} \pm 1} \quad (5.23)$$

where β is the Hawking temperature [2], ω is the frequency (energy) of the radiation quantum, and $+$ ($-$) corresponds to fermions (bosons) respectively. The full spectrum of the (Fourier-analyzed) radiation is encoded in the higher moments or *fluxes* [41]. The energy flux, for instance, is given by the second moment of $N^\pm(\omega)$, $F_2(\omega) = \int_0^\infty \frac{d\omega}{2\pi} \omega N^\pm(\omega)$. The complete thermal Hawking radiation spectrum is specified by the infinity of higher moments:

$$F_{2n}^+ = \int_0^\infty \frac{d\omega}{2\pi} \omega^{n-1} N^+(\omega) = (1 - 2^{1-2n}) \frac{B_{2n}}{8\pi n} \kappa^{2n}, \quad (5.24)$$

or

$$F_{2n}^- = \int_0^\infty \frac{d\omega}{2\pi} \omega^{n-1} N^-(\omega) = \frac{B_{2n}}{8\pi n} \kappa^{2n}, \quad (5.25)$$

where the B_{2n} are the Bernoulli numbers and $\kappa = 2\pi/\beta$ is the surface gravity of the black hole.

An interesting proposal was made in [41] that the higher fluxes F_{2n} , $n > 1$ could be connected to phenomenological higher-spin currents (with appropriate normal ordering), *i.e.* higher-spin generalizations of the energy-momentum tensor, which is the spin-two current associated with the energy flux. As shown in [41], these higher currents can be expressed in terms of two dimensional boson and fermion fields, involving linear combinations of (two-dimensional) space-time derivatives acting on the fields. In an appropriate light-cone coordinate (u, v) description, the outgoing near-horizon Hawking radiation is described by holomorphic (u -dependent) currents of the following form in the scalar case [41]:

$$J_{uu\dots u}^B = \text{linear combinations of } : (-1)^{n+m} \partial_u^m \phi \partial_u^{2n-m} \phi : , \quad (5.26)$$

where $: \dots :$ denotes appropriate normal ordering, as defined in [41]⁹.

It should be noted that there are ambiguities in the representation of the higher moments of the Hawking radiation in terms of conformal fields on the horizon, reflected in the relative coefficients of the various terms appearing in the holomorphic currents (5.26). Moreover, the currents are not normalizable in general. However, these issues have been resolved by the proposal of [42] that the coefficients of these currents be fixed by a symmetry principle, namely by the postulate that there is a higher symmetry in the horizon of the black hole than the Virasoro symmetry, namely a W_∞ algebra. It was to be expected from their higher-derivative holomorphic structure, $\partial_z^n \chi$ ($\chi = \phi, \psi$) [24] that the currents could be cast in such a way as to form an infinite-dimensional algebra.

In a flat two-dimensional space-time, after *Euclideanisation* and replacing the light-cone coordinates u, v by the complex variables z, \bar{z} respectively, the w_∞ -generating bosonic currents for the conformal spin s states can be written in the form

$$j_{z \dots z}^{(s)B} = q^{s-2} \frac{2^{s-3} s!}{(2s-3)!!} \sum_{k=1}^{s-1} (-1)^k \left[\frac{1}{s-1} (s) - 1k (s) - 1s - k \right] : \partial_z^k \phi(z) \partial_z^{s-k} \bar{\phi}(z), \quad (5.27)$$

where $: \dots :$ denotes normal ordering, defined in [42], and q is a complex (in general) deformation parameter [24]. In this formalism the holomorphic free fields $\phi(z)$ are assumed to have two-point functions of the form $\langle \phi(z) \bar{\phi}(z') \rangle = -\ln(z - z')$, with the others vanishing. The deformation parameter q can be fixed [42] by demanding that the currents (5.27), when *covariantised* as appropriate for the curved space-time of the spherical symmetric black hole (which is only conformally equivalent to a flat space), reproduce the higher moments of the Hawking flux. The $s = 2$ current is independent of the q deformation parameter, as expected from the fact that this current can be identified unambiguously with the holomorphic stress tensor

$$j_{uu}^{(2)} = -2\pi T_{uu}^{\text{hol}}.$$

However, the higher-spin currents depend on q , and one must fix $q = -i/4$ [42] in order to reproduce the Hawking fluxes for bosons (5.25) when covariantising the expression (5.27) by replacing the ordinary derivatives with covariant ones, as is appropriate for discussing the Hawking flux in the conformally-flat metric that represents the near-horizon geometry of the black hole.

As discussed in great detail in [42], the currents (5.27) and their fermionic counterparts with spins higher than two are free of conformal or diffeomorphism anomalies (or, if the latter exist, they are trivial). This is consistent with the fact that the higher moments of the Hawking radiation are expected to describe a theory free of gravitational anomalies, since only the spin-two current (stress tensor) of the theory has diffeomorphism or conformal anomalies, and their cancellation required the appearance of the Hawking radiation spectrum [43]. Moreover, this is also consistent with the fact that, if these currents had conformal anomalies, they would correspond to new (non-gauge) quantum numbers of the black hole, in violation of the no-hair theorem. Hawking radiation is consistent with this theorem, which is then reflected in the absence of anomalies in the currents corresponding to currents with spins higher than two.

⁹For the fermion case, see [41] and [44].

The covariant higher-spin currents $J_{\mu_1 \mu_2 \dots \mu_n}^{(s)B,F}$ are sourced by appropriate background fields $\mathcal{B}_{\mu_1 \mu_2 \dots \mu_n}^{(s)B,F}$:

$$J_{\mu_1 \dots \mu_n}^{(s)B,F} = \frac{1}{\sqrt{g}} \frac{\delta}{\delta \mathcal{B}_{\mu_1 \dots \mu_n}^{(s)B,F}} S, \quad (5.28)$$

where S is the two-dimensional effective action of the Hawking radiation in the near-horizon geometry of the spherically-symmetric black hole. The relevant interactions in this effective geometry are then simply given by

$$S_{\text{int}} = \int_{\text{near horizon 2D space-time}} d^2x \sqrt{g} \sum_{\alpha=B,F} \mathcal{B}^{(s)\alpha \mu_1 \dots \mu_n} J_{\mu_1 \dots \mu_n}^{(s)\alpha}, \quad (5.29)$$

and the background fields $\mathcal{B}^{(s)\alpha \mu_1 \dots \mu_n}$ may be taken to vanish at asymptotic spatial infinity. Eqn. (5.29) generalises the spin-2 case, in which the corresponding spin-2 current (the stress tensor) couples to the graviton field, $\int d^2x \sqrt{g} T^{\mu\nu} g_{\mu\nu}$, which is characterised by diffeomorphism invariance (acting as a ‘gauge symmetry’): $\delta g_{\mu\nu} = \partial_{(\mu} \xi_{\nu)}$ for an infinitesimal diffeomorphism $\xi_\mu \rightarrow x_\mu + \xi_\mu$, provided the stress tensor is conserved¹⁰. As already remarked, the higher-spin currents are free from conformal and diffeomorphism anomalies [42] and are conserved exactly, and their conservation is associated with an infinity of Abelian gauge symmetries of the form

$$\mathcal{B}_{\mu_1 \dots \mu_n}^{(s)} \rightarrow \mathcal{B}_{\mu_1 \dots \mu_n}^{(s)} + \partial_{(\mu_1} \Xi_{\mu_2 \dots \mu_n)}, \quad (5.30)$$

where the (\dots) indicate the appropriate symmetrization of indices. The presence of these infinite gauge symmetries is consistent with the no-hair theorem, as the spatial integrals of the currents correspond to conserved charges. The existence of a W_∞ symmetry of the matter in the near-horizon geometry, which is larger than the Virasoro algebra, results in the complete integrability of the matter system, and is analogous to the case of matter in the near-horizon geometries of black-hole structures in the context of string theory [10, 9], discussed above.

This W_∞ algebra is *phase-space-preserving*, as are the W_∞ algebras discussed in the stringy cases above. To see this, one may rewrite the (traceless) energy-momentum tensor of the two-dimensional effective scalar theory using a point-splitting method [42]:

$$\begin{aligned} T_{\mu\nu} &= \lim_{y \rightarrow 0} \partial_\mu \phi(x-y) \partial_\nu \phi(x+y) - g_{\mu\nu} (\text{stress} - \text{tensor trace}) \\ &= \sum_{i=0} \sum_{j=0} \frac{(-1)^i}{i! j!} : y^{\mu_1} \dots y^{\mu_i} y^{\nu_1} \dots y^{\nu_j} \partial_\mu \partial_{\mu_1} \dots \partial_{\mu_i} \phi(x) \partial_\nu \partial_{\nu_1} \dots \partial_{\nu_j} \phi(x) : \end{aligned} \quad (5.31)$$

This expression is covariantised by replacing the partial derivatives by covariant derivatives, giving the right-hand-side of (5.31) a complicated expansion in terms of products of the higher-spin currents with y -dependent factors (5.31) that correspond via complicated background tensors $\mathcal{B}_{\mu_1 \dots \mu_n}^{(s)}$ to the aforementioned background fields that source the higher-spin currents. For our purposes, the most important feature of (5.31) is the fact that the right-hand side depends not only on the coordinate x^μ but also on the quantity $y^\mu = dx^\mu$ in the cotangent bundle, and thus lives in a symplectic

¹⁰In the black-hole case, as we have discussed above, the diffeomorphism invariance is broken by the outgoing flux, but the form of the transformation is included in (5.30).

phase-space manifold, showing that the $W_{1+\infty}$ algebra generated by the higher-spin currents of the Hawking radiation spectrum is a phase-space algebra.

One may associate these symmetries classically with *horizon-area-preserving diffeomorphisms*, following the discussion of Section 5.1 for the $SU(\infty)$ coloured black hole case, but we reserve details for a future publication [44]. For our purposes here we only mention that the above-mentioned $W_{1+\infty}$ field theories on the black-hole horizon (for fermion or boson fields) may be “gauged”, in the way discussed in [25] and reviewed in Section 5.3, by considering the extension of the fields in a space augmented with two extra dimensions $\xi, \bar{\xi}$ that can be taken to be the coordinates of the spherical horizon surface. The corresponding field theories of these $(d+2)$ -dimensional fields, where d the target-space dimensionality of the field theories on the horizon, are gauged in the way discussed in detail in [25]¹¹ and can be shown to be invariant under the phase-space-area-preserving diffeomorphisms of the coordinates $\xi, \bar{\xi}$, which can be taken to represent the spherical black-hole horizon. It is in this way that the infinite-dimensional dynamical phase-space-area-preserving $W_{1+\infty}$ symmetries (5.31) of the near-horizon currents representing the Hawking radiation spectrum are related to classical horizon-preserving w_∞ symmetry algebras. This is consistent with the identification of the classical black hole entropy (that is proportional to the area) with a classically-conserved Noether charge [27].

Before closing this discussion, we note an important difference of the two-dimensional $c = 1$ string theory description [4, 6] of the black-hole singularity [10, 12] from a generic two-dimensional field-theoretic representation of the outgoing Hawking radiation, as discussed above. In the latter case, as we have seen, higher-spin currents consist of higher derivatives of propagating local (boson or fermion) fields, whereas in the $c = 1$ string theory representation one encounters *necessarily* non-propagating discrete delocalised states in the excitation spectrum, which are non-thermal. These states exist over and above the ordinary Hawking radiation fields and, as we have discussed above, their presence has highly non-trivial consequences. These delocalised states correspond to gauge states in the $c = 1$ string theory [39], which carry the phase-space-area-conserving and coherence-preserving W_∞ charges of the two-dimensional black hole singularity.

In our treatment, the singularity of the four-dimensional spherically-symmetric stringy black hole is characterised by an infinite-dimensional $W_{1+\infty}$ phase-space symmetry, whose charges are carried by the (infinity of) delocalised discrete, non-propagating higher-spin gauge states that are responsible for maintaining quantum coherence [10]. On the other hand, the black-hole horizon carries another set of phase-space $W_{1+\infty}$ symmetries, corresponding to higher-spin currents composed of the propagating modes of the two-dimensional effective field theory in the near-horizon geometry, representing the outgoing Hawking radiation flux. Thus, the Hawking radiation W_∞ can be made to preserve the horizon area, but it is the discrete non-propagating string states that preserve quantum coherence.

¹¹One has $d = 1$ for the holomorphic fields that represent the outgoing Hawking radiation [43, 41, 42], which in the case of string-theory-inspired black holes [10] can be represented non-perturbatively via the $c = 1$ matrix model [40].

6. Quo Vadis Supertranslations?

6.1 D-Brane Recoil and Supertranslations

In order to understand better the microscopic mechanism for encoding information on the horizon, we consider the back-reaction of the black-hole horizon induced by its interaction with infalling matter. We first concentrate on the two-dimensional stringy black hole where the horizon is a point in space. As discussed in the previous section, we can represent the horizon of such a black hole as a D-particle defect in space, whereas the horizon of a four-dimensional black hole can be represented by a spherical Dirichlet brane. Using D-branes enables us to consider the ‘momentary’ capture of string matter by the horizon, in the sense of a spontaneous change of world-sheet boundary conditions from Neuman to Dirichlet. Such a process will lead to splitting of a closed-string state into two open ones. If one considers closed-string states as corresponding to gravitons and open-string states to gauge particles (including photons), this process may correspond to the conversion of an infalling graviton into a pair of photons. In general, when one represents the horizon of a black hole as a D-brane, the interaction with infalling string states implies a ‘recoil’ of the D-brane.

In the pilot case of a black hole in two target-space-time dimensions, we consider a matter particle represented by an open string falling into the horizon the interaction of the string with the D-particle horizon implies that at least one end of the open string attaches to the D-particle defect. As a result of the interaction, the D-particle undergoes a non-trivial change in velocity

$$u_r = \frac{g_s}{M_s} \Delta p_r = \frac{g_s}{M_s} \xi_r p_r, \quad (6.1)$$

where ξ_r denotes the fraction of the incident matter particle momentum that corresponds to the momentum transfer Δp_r during the scattering. As discussed in [36] the non-trivial capture and splitting of the open string during its interaction with the D-particle, and the recoil of the latter, result in a *local* effective metric distortion of the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu : \quad h_{0r} = u_r. \quad (6.2)$$

In the black hole case of [4] with dilaton hair: $\Phi = -2 \ln \cosh r$, the string coupling $g_s = e^\Phi$ becomes weak at large distances: $g_s \rightarrow 0$ for $r \rightarrow \infty$. Hence $u_i \rightarrow 0$ and the space-time distortion vanishes at large distances, where the space is asymptotically flat.

The metric (6.2) can be generalised to higher dimensions, with a D-brane horizon recoiling along the i ’th spatial dimension, in which case the space-time distortion due to the recoiling D-brane horizon can be written as

$$ds^2 = dt^2 + 2u_i dx^i dt - \delta_{ij} dx^i dx^j. \quad (6.3)$$

This metric was determined from world-sheet (logarithmic) conformal field theory considerations in [36, 35]: the world-sheet deformations representing the recoil of the D-brane close a logarithmic conformal algebra on the world-sheet of the string that represents a dragging of the reference frame by the D-brane horizon as it moves slowly on the flat space-time background. On the other hand, the string excitations represent relativistic particles, and thus they move at the local speed of light.

One may perform a time coordinate change in the metric (6.3) to write it in the following form, which is valid up to terms u^3 for small recoil velocities $|\vec{u}| \ll 1$, as is appropriate for macroscopic black holes:

$$ds^2 = dt_{\text{ff}}^2 + 2u_i dx^i dt_{\text{ff}} - \delta_{ij}(dx^i - u^i dt_{\text{ff}})(dx^j - u^j dt_{\text{ff}}) + \mathcal{O}(u^3). \quad (6.4)$$

The metric (6.4) is the Gullstrand-Painlevé metric [45], which represents the geometry around the exterior of a Schwarzschild black hole. It represents the space falling into the black hole as a Gallilean ‘river’ on a flat space-time in which relativistic ‘fish’ may swim. The river represents the frame of the recoiling D-particle horizon, while the fish are the relativistic matter strings [46]. In (6.4), t_{ff} is the time of a free-floating observer who is at rest at infinity, compared to the centre of the black hole. We stress that, in the case of a black hole, the relative velocities u^i are coordinate-dependent, as already mentioned, due to the variation in the string coupling from being strong near the black hole singularity to being weak on the horizon.

In general, the recoil velocity has components normal and tangential to the horizon. The former can be associated with changes in the horizon area, and hence the black-hole entropy, whereas the latter would not change the area. In case of such a tangential recoil component, metrics of the form (6.4), written in Bondi coordinates, are of the same form as metrics that have been discussed in the past in connection with gravitational wave radiation in asymptotically-flat regions of space time [16, 17], and are known to be associated with supertranslations of the Bondi retarded time $u \equiv t_{\text{ff}} + r$

$$u \rightarrow u + \alpha(\theta, \phi), \quad (6.5)$$

where $\alpha(\theta, \phi)$ is a function of the angular coordinates θ, ϕ . Such a retarded time was used in [16, 17] to discuss outgoing gravitational wave signals arriving at a distant observation point. Such BBMS⁺ transformations form an infinite-dimensional set of diffeomorphisms that include as a subgroup the four-parameter group of ordinary translations. In the case of matter falling into the black-hole horizon one may use instead the BBMS[−] transformations pertaining to the advanced time $v = t_{\text{ff}} - r$, which amount to the supertranslations

$$v \rightarrow v + \xi(\theta, \phi), \quad (6.6)$$

where $\xi(\theta, \phi)$ is a function of the angular coordinates θ, ϕ on the black-hole horizon brane. The retarded (or advanced) time is viewed in general as a scalar function of the coordinates $u(x^\mu)$ that obeys $u_{,\mu} u_{,\nu} g^{\mu\nu} = 0$, implying that the hypersurfaces $u = \text{constant}$ are light-like.

The generic space-times on which there are supertranslations that leave invariant the boundary conditions have the form [17]

$$ds^2 = \left(\frac{V}{r} e^{2\beta}\right) du^2 - 2e^{2\beta} du dr + r^2 h_{AB} (dx^A - U^A du)(dx^B - U^B du), \\ 2h_{AB} dx^A dx^B = (e^{2\beta} + e^{2\gamma}) d\theta^2 + 4\sin\theta \sinh(\gamma - \delta) d\theta d\phi + \sin^2\theta (e^{-2\beta} + e^{-2\gamma}) d\phi^2, \quad (6.7)$$

where u is a retarded time and (r, x^A) are the three spatial coordinates, with $x^A = (\theta, \phi)$ the angular variables of the four-dimensional space-time appropriate for spherically-symmetric solutions of the gravitational equations, with determinant $\det h_{AB} = \sin^2\theta$. The functions $V, U^A, \beta, \gamma, \delta$ are arbitrary functions of the coordinates and, to match them with (6.4), one performs the advanced (retarded) time transformation $u(v)$ from t_{ff} as mentioned previously. The asymptotic flatness of (6.4), due to

the asymptotic vanishing of the recoil-induced distortion of the space-time surrounding the black hole horizon, implies the boundary conditions $\text{Lim} \left(V/r = 1, \text{Lim} (rU^A) = \beta = \gamma = \delta \right) = 0$ for fixed u or v and $r \rightarrow \infty$.

6.2 Supertranslations are not Enough

Hawking [18] has suggested that such supertranslations of an *advanced* Bondi time on the black-hole horizon may solve the issue of quantum coherence. The proposal builds upon analyses in the asymptotically-flat regime of a generic Schwarzschild black hole space-time by Strominger and collaborators [14, 15], who have provided arguments that the horizon supertranslation transformations can be viewed as a conformal Kac-Moody symmetry group, entailing an infinity of conservation laws. This symmetry group is an infinite-dimensional diffeomorphism group that leaves invariant the asymptotic BMS states as well as the generic quantum gravity scattering matrix defined by means of asymptotic *in* and *out* quantum states. Section 6 of the first reference in [15] made an association of these infinite supertranslation charges with hair for black holes, and suggested that such charges ‘may bear on the information puzzle’¹².

This suggestion looks similar in nature to the one we described above and in our previous works [10, 12], but there are important differences, and we do not think that supertranslations are sufficient to retain quantum coherence.

The supertranslations (6.5), (6.6) are symmetries of the asymptotic (large r) metrics and not of the full black hole background space-time, since they correspond to physically-inequivalent metrics, as discussed in [16, 17]. This may be viewed as a sort of ‘spontaneous breaking’ of the supertranslation symmetries of the asymptotic Minkowski space-time by the black hole background. The associated Goldstone bosons have been identified tentatively (in the semiclassical black-hole limit of infinite entropy) [19] with delocalised graviton states of infinite wavelength. This proposal may sound similar to our topological stringy states, though the situation for a finite-area (finite-entropy) black hole is far from clear.

However, in our picture the discrete topological string states include such soft graviton states as a subset, along with an infinity of other higher-spin discrete (topological) states, all corresponding to conserved charges. These higher-spin states are separated by mass gaps $\propto nM_s/g_s$, $n \in \mathbb{Z}^+$. Hence, given that on the horizon the string coupling is weak, such states may *seem* to be decoupled at the level of the local effective field theory level (LEFT), leaving only the massless graviton states as relevant. However, we conjecture that, for information retention at a *finite-area* black-hole horizon, one must consider the entirety of the delocalised (topological) higher-spin states, whose treatment goes beyond LEFT. These realize an infinite-dimensional *area-preserving* quantum W_∞ symmetry algebra (with its infinity of conserved charges) in the coset black-hole model. This

¹²We know no obvious association of Kac-Moody algebras to the W_∞ area-preserving algebras discussed here, except in the supersymmetric case considered in [47], where an area-preserving diffeomorphism algebra $\text{SDiff}(\mathcal{M})$ of a two-dimensional surface \mathcal{M} acts as a derivation algebra on a super-Kac-Moody algebra, much as Virasoro algebras act as derivation algebras on Kac-Moody algebras on a one-dimensional circle S^1 . The implications of this result in our case are not obvious. However, we recall that when one supersymmetrises the two-dimensional black hole to a (twisted topological) $N = 2$ theory, a double $W_{1+\infty} \otimes W_{1+\infty}$ describes the singularity structure [48], which is broken $\rightarrow W_{1+\infty}$ away from it. The findings of [47] may be relevant in the case when this $N = 2$ model is embedded in four dimensions. However, any connection between the supertranslation $U(1)$ Kac-Moody algebra on the black-hole horizon [14] with (super-) W_∞ horizon-area-preserving diffeomorphisms is currently unclear.

picture is in agreement with the representation of the two-dimensional black hole horizon in four space-time dimensions as the world-sheet of a string, where a classical w_∞ world-sheet symmetry preserves the area, and thus ensures information retention. Such world-sheet w_∞ symmetries can be elevated to target space by means of appropriate world-sheet deformations, corresponding to the various excitations of the higher-spin target-space states [13]. Moreover, we have argued that a recoil displacement of a D-brane due to a ‘sudden’ impulse, as is the case of a string splitting on the horizon brane, merely mislays information that is stored on the horizon [37], in the sense that entanglement is induced between the recoiling quantum D-brane horizon and the quantum matter subsystem. The precise reproduction of the finite-area black-hole entropy for the generic four-dimensional black-hole case remains to be worked out (although arguments have been given in this direction for the two-dimensional stringy black hole in [12]), and we plan to return to this issue in the near future.

We consider that our view of discrete W -hair and the representation of the black-hole horizon as a D(irichlet) brane recoiling under the interaction with infalling matter is more appropriate than supertranslation invariance for maintaining quantum coherence. We would argue that any approach based on general relativity alone is limited in scope, and that string-theoretical considerations based on the rigorous counting of black-hole microstates, such as those outlined above, constitute a much more promising approach to the black-hole information-loss problem. In particular, we recall that the W_∞ algebra corresponds to an $SU(N \rightarrow \infty)$ algebra, as discussed in Section 4.4, and that the supertranslations (and the corresponding $U(1)$ gauge states) do not include candidates for all the infinitely-coloured hair discussed there.

7. Conclusions and Outlook

We have reviewed and extended in this paper our previous arguments for the importance of infinite-dimensional W_∞ symmetry in information retention by stringy black holes. This symmetry preserves two-dimensional area. As such, it plays a key rôle at the world-sheet level, ‘balancing the information books’ [10] in the two-dimensional black-hole model of [4], which may be elevated to four dimensions to describe a spherically-symmetric black hole [12]. In this case, the entropy is proportional to the area of the horizon, so entropy is therefore clearly conserved by a W_∞ symmetry. The same symmetry preserves the area of the two-dimensional phase space describing a fermion interacting with a four-dimensional extremal black hole in the context of $N = 2$, $D = 4$ supergravity [9].

As we have stressed in this paper, there are still many open issues in our approach to the black-hole information problem based on W_∞ symmetry, but we consider it to be much more complete and promising than recent suggestions [14, 15] based on supertranslations [16]. The W_∞ symmetry is firmly embedded within string theory, which must surely be taken into account in any resolution of the black-hole information problem. Also, as can be seen from its relation to $SU(\infty)$, W_∞ symmetry certainly offers conserved charges (W -hair) that are not provided by supertranslations.

We have argued elsewhere that the W -hair of a four-dimensional black hole is in principle measurable [13], though this may not be feasible in practice [11]. For this reason, we consider that black holes ‘mislays’ information rather than ‘lose’ it. However, we still lack a specific ‘Ariadne’s

thread’ of external measurements that can enable us to reconstruct the information ‘misaid’ within the black-hole labyrinth.

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